Decidable Weighted Expressions with Presburger Combinators

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Boolean vs Quantitative Languages

$$L : \Sigma^* \to \{0, 1\}$$

Classical decision problems

Emptiness	$\exists u. f(u) \geq 1$
Universality	$\forall u. f(u) \geq 1$
Inclusion	$\forall u. f(u) \geq g(u)$
Equivalence	$\forall u. f(u) = g(u)$

Boolean vs Quantitative Languages

$$L : \Sigma^* \to \cancel{\{0, 1\}}\ \mathbb{Z} \cup \{-\infty\}$$

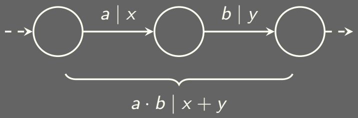
Classical quantitative decision problems

Emptiness	$\exists u. f(u) \geq \cancel{1}\ \nu$	for some threshold ν
Universality	$\forall u. f(u) \geq \cancel{1}\ \nu$	for some threshold ν
Inclusion	$\forall u. f(u) \geq g(u)$	
Equivalence	$\forall u. f(u) = g(u)$	

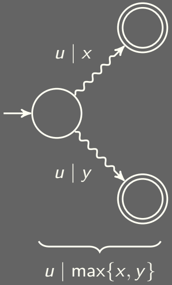
Classical Model: Weighted Automata

$(\max,+)$ WA

Transition sequence



Non-determinism



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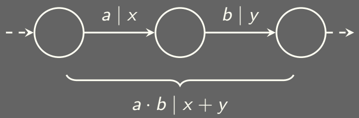

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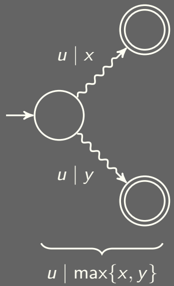


Undecidability [Krob 1994]

Quantitative language-inclusion is undecidable for (max,+) WA

- Even for linearly ambiguous automata [Colcombet 2010]

Decidable Formalisms: Restriction

Finitely ambiguous (max,+) WA [Filiot et al. 2012]

Define functions of the form,

$$u \mapsto \max\{\mathcal{A}_1(u), \ldots, \mathcal{A}_k(u)\}$$

$\mathcal{A}_i$: Unambiguous WA

- ☺ Quantitative decision problems are DECIDABLE
- ☺ Closed under *max* and *sum*
- ☹ Limited expressive power (*min*, *minus*, ...)

Decidable Formalisms: New model

Mean-payoff expressions [Chatterjee et al. 2010]

$$E ::= \mathcal{A} \mid \max(E, E) \mid \min(E, E) \mid E + E \mid -E$$

$\mathcal{A}$: Deterministic WA

- ☺ Quantitative decision problems are PSPACE-COMPLETE [Velner 2012]
- ☺ Closed under *max*, *min*, *sum* and *minus*
- ☹ Determinism (define Lipschitz continuous functions)
- ☹ Does **not** contain all finitely ambiguous (max,+) WA
- ☹ Monolithism (apply on the whole word)

Contributions

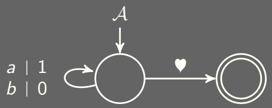
❶ Simple expressions

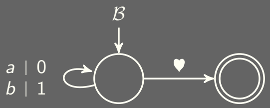
$$E ::= \mathcal{A} \mid \phi(E, E)$$

$\mathcal{A}$: Unambiguous WA

ϕ : $\exists$FO$[\leq, +, 0, 1]$ formula defining function with arity two

Example





$$E = \max(\mathcal{A} - \mathcal{B}, \mathcal{B} - \mathcal{A})$$

$$u \mapsto |\mathcal{A}(u) - \mathcal{B}(u)|$$

Contributions

➊ Simple expressions

$$E ::= \mathcal{A} \mid \phi(E, E)$$

$\mathcal{A}$: Unambiguous WA

ϕ : $\exists$FO[$\leq, +, 0, 1$] formula defining function with arity two

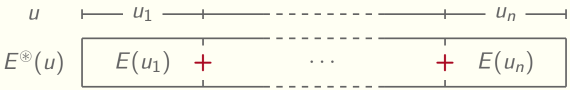
- ☺ Quantitative decision problems are PSpace-Complete
- ☺ Closed under Presburger definable functions
- ☺ Contain all finitely ambiguous (max,+) WA
- ☹ Monolithism (apply on the whole word)

Contributions

❷ Iterable expressions

$$E ::= \mathcal{A} \mid \phi(E, E) \mid E^{\circledast}$$

- Sum arbitrarily many factors
- Unique decomposition required



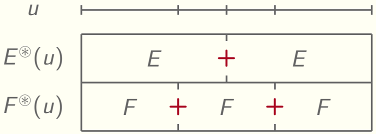
Examples

$E^{\circledast}$	$u_1 \heartsuit\, u_2 \heartsuit \ldots u_n \heartsuit \mapsto \sum_{i=1}^{n} E(u_i)$
$\phi(E^{\circledast}, F^{\circledast})$	$u \mapsto \phi \left\{ \sum_{i=1}^{n} E(u_i) \,,\, \sum_{j=1}^{m} F(v_i) \right\}$

Results

Theorem (Iterable Expressions)

Quantitative decision problems are UNDECIDABLE



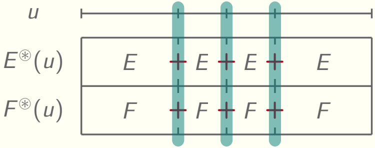
Results

Theorem (Iterable Expressions)

Quantitative decision problems are UNDECIDABLE

Theorem (Synchronised Iterable Expressions)

Quantitative decision problems are DECIDABLE



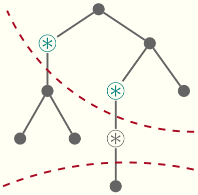
Results

Theorem (Iterable Expressions)

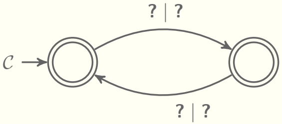
Quantitative decision problems are UNDECIDABLE

Theorem (Synchronised Iterable Expressions)

Quantitative decision problems are DECIDABLE
Synchronisation property is PTIME



Weighted Chop Automata

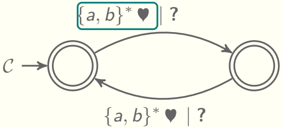


New model

- Generalise unambiguous WA
- Recursive definition

Weighted Chop Automata

Regular language



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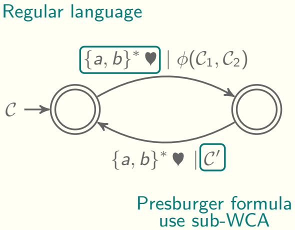
Weighted Chop Automata



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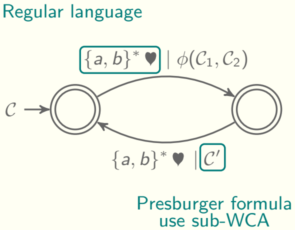
Weighted Chop Automata



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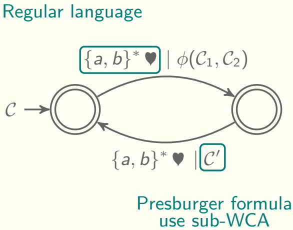
Example

$\mathcal{C}(aab\heartsuit\, baa\heartsuit) =$
$\phi\left(\mathcal{C}_1(aab\heartsuit), \mathcal{C}_2(aab\heartsuit)\right) + \mathcal{C}'(baa\heartsuit)$

Weighted Chop Automata



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Operators for expressiveness equivalence

$$\left.\begin{array}{l} E \odot F : u_1 u_2 \mapsto E(u_1) + F(u_2) \\ E \triangleright F : u \mapsto \textbf{if } u \in \mathrm{dom}(E) \textbf{ then } E(u) \textbf{ else } F(u) \end{array}\right\} \text{[Alur 2014]}$$

Conclusion

Summary

Simple expressions: PSpace-Complete
Sum-iterable expressions: Undecidable
Synchronised sum-iterable expressions: Decidable

Perspective

Iterate other operations (*max*, Presburger definable functions, …)

Thanks!

Conclusion

Summary

Simple expressions: PSPACE-COMPLETE
Sum-iterable expressions: UNDECIDABLE
Synchronised sum-iterable expressions: DECIDABLE

Perspective

Iterate other operations (*max*, Presburger definable functions, ...)

$$\begin{array}{cccc} r\,r\,r\,r\,g & r\,r\,r\,r\,r\,r\,r\,g & \ldots & r\,r\,r\,r\,r\,g \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$\underbrace{4 \qquad 7 \qquad r^n g \mapsto n \qquad 5}_{\text{iterate max}}$$

Thanks!